

Background Field Equations for the Duality Symmetric String

David S. Berman¹, Neil B. Copland² and Daniel C. Thompson³

Queen Mary College, University of London,
Department of Physics,
Mile End Road,
London, E1 4NS, England

Abstract

This paper describes the background field equations for strings in T-duality symmetric formalisms in which the dimension of target space is doubled and the sigma model supplemented with constraints. These are calculated by demanding the vanishing of the beta-functional of the sigma model couplings in the doubled target space. We demonstrate the equivalence with the background field equations of the standard string sigma model.

¹email: D.S.Berman@qmul.ac.uk

²email: N.B.Copland@qmul.ac.uk

³email: D.C.Thompson@qmul.ac.uk

1 Introduction

T-duality is one of the cornerstones of string theory. It is an intrinsically stringy effect which relates small to large manifolds exchanging winding with momentum modes. From a space-time perspective T-duality is a solution generating symmetry of the low energy equations of motion but from the world sheet point of view, T-duality is a non-perturbative symmetry. Importantly, the presence of T-duality allows for the construction of non-geometric manifolds where locally geometric regions are patched together with T-duality transformations. Such constructions, known as T-folds [1], may play a crucial role in moduli stabilisation and certainly are an important part of any string landscape.

Given the importance of T-duality it is desirable that this symmetry is made manifest in the string sigma model. There have been various attempts in the past to develop a formalism where T-duality is a symmetry of the action [2]. We will use the form most recently championed by Hull [3] as our starting point. This doubles the dimension of the target space with the doubled dimensions being related by T-duality. Additional constraints are then needed to reduce the degrees of freedom. These take the form of a set of chirality constraints. The result is that the formalism has manifest T-duality with a doubled target space. The key difficulty is dealing with the constraints in the correct way. Demonstrating the classical equivalence of this formalism to the usual sigma model is straightforward, however, showing quantum equivalence is less trivial. This has been discussed in [7–9].

In this paper we would like to examine the beta-functional of the string in this formalism so as to determine the background-field equations for the doubled space. This project was prompted by various questions. The most important was to determine if the background-field equations arising from the one-loop beta-functional for the doubled formalism were the same as for the usual string. A priori this did not have to be the case. In fact, one may imagine that they will be different since the doubled formalism naturally incorporates the string winding modes which could in principle correct the usual the beta-function. We know that world sheet instantons

correct T-duality [10,11] and so since the doubled space contains the naive T-dual one may think of all sorts of possibilities that could arise for corrections to the doubled geometry.

We begin by briefly introducing the doubled formalism before incorporating the constraints into the action so that we can use the background-field method. We then carry out the background-field expansion to find the resulting one loop beta-functional for the background doubled metric. Finally the relation to the non-doubled geometry is described. This shows that the constraints on the doubled geometry required for conformal invariance are equivalent to the usual background-field equations for the standard string sigma model.

2 The doubled formalism

The doubled formalism [1,3,4] is an alternate description of string theory on target spaces that are locally T^n bundles, with fibre coordinates X^i , over a base N with coordinates Y^a . The fibre is doubled to be a T^{2n} with $2n$ coordinates \mathbb{X}^A . The doubled sigma model then has Lagrangian⁴

$$\mathcal{L} = \frac{1}{4} \mathcal{H}_{AB}(Y) d\mathbb{X}^A \wedge *d\mathbb{X}^B + \mathcal{L}(Y) + \mathcal{L}_{top}(\mathbb{X}) \quad (1)$$

where $\mathcal{L}(Y)$ is the standard Lagrangian for a string on the base and $\mathcal{H}(Y)$ is a metric on the fibre⁵. \mathcal{L}_{top} is a purely topological term described in [4]. It is vital to obtaining the equivalence of the doubled to the non-doubled partition functions [7] but it will play no role in the beta-functions and so will be dropped from now on. One may choose a frame where the metric \mathcal{H} has an $O(n, n)/O(n) \times O(n)$ coset form as follows:

$$\mathcal{H}_{AB}(Y) = \begin{pmatrix} h - bh^{-1}b & bh^{-1} \\ -h^{-1}b & h^{-1} \end{pmatrix}. \quad (2)$$

⁴The complete formalism also introduces a 1-form connection for the fibration $\mathcal{A}^A(Y)$ which we set to zero throughout this paper.

⁵We adopt the conventions of [4]; the worldsheet signature is $(+, -)$, $\partial_{\pm} = \partial_0 \pm \partial_1$, $\epsilon_{01} = 1 = \epsilon^{01}$ and for convenience we have dropped an overall factor of 2π . The factor of $\frac{1}{4}$ in (1) is half the usual normalisation and is required to make contact with the standard sigma model.

h and b are the target space metric and B -field on the fibre of the undoubled space. In this frame $\mathbb{X}^A = (X^i, \tilde{X}_i)$ with $\{\tilde{X}_i\}$ the coordinates on the T-dual torus. We must then supplement this doubled sigma model action with a set of constraints so as to obtain the correct number of degrees of freedom and be equivalent to the usual non-doubled sigma model. The constraints are given by

$$d\mathbb{X}^A = L^{AB} \mathcal{H}_{BC} * d\mathbb{X}^C, \quad (3)$$

where the L is an $O(n, n)$ invariant metric that can be used to raise and lower indices on \mathcal{H} which in this basis is given by

$$L_{AB} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (4)$$

To understand this constraint it is helpful to introduce a vielbein to allow a change to a chiral frame (denoted by barred indices) where:

$$\mathcal{H}_{\bar{A}\bar{B}}(y) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad L_{\bar{A}\bar{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (5)$$

In this frame the constraint (3) is a chirality constraint ensuring that half the $\mathbb{X}^{\bar{A}}$ are chiral Bosons and half are anti-chiral Bosons.

As a simple example let us consider the $n = 1$ case ie. a circle, with constant radius R . The doubled action on the fibre is

$$S_d = \frac{1}{4} R^2 \int dX \wedge *dX + \frac{1}{4} R^{-2} \int d\tilde{X} \wedge *d\tilde{X}. \quad (6)$$

As in [7], we make the change to basis in which the fields are chiral:

$$P = RX + R^{-1} \tilde{X}, \quad \partial_- P = 0, \quad (7)$$

$$Q = RX - R^{-1} \tilde{X}, \quad \partial_+ Q = 0. \quad (8)$$

In this basis the action becomes

$$S_d = \frac{1}{8} \int dP \wedge *dP + \frac{1}{8} \int dQ \wedge *dQ. \quad (9)$$

One may then incorporate the constraints into the action using the method of Pasti, Sorokin and Tonin [12]. We define one-forms

$$\mathcal{P} = dP - *dP, \quad \mathcal{Q} = dQ + *dQ, \quad (10)$$

which vanish on the constraint. These allow us to incorporate the constraint into the action via the introduction of two auxiliary closed one-forms u and v as follows:

$$S_{PST} = \frac{1}{8} \int dP \wedge *dP + \frac{1}{8} \int dQ \wedge *dQ - \frac{1}{8} \int d^2\sigma \left(\frac{(\mathcal{P}_m u^m)^2}{u^2} + \frac{(\mathcal{Q}_m v^m)^2}{v^2} \right). \quad (11)$$

The PST action works by essentially introducing a new gauge symmetry, the PST symmetry, that allows the gauging away of fields that do not obey the chiral constraints. Thus only the fields obeying the chiral constraints are physical.

There are now two ways to proceed. One may either gauge fix the PST-style action immediately which will break manifest Lorentz invariance or try to quantise covariantly and introduce ghosts to deal with the PST gauge symmetry. In this paper we choose the non-covariant option and immediately gauge fix to give a Floreanini-Jackiw [13] style action. Picking the auxiliary PST fields (u and v) to be time-like produces two copies of the FJ action (one chiral and one anti-chiral)

$$S = \frac{1}{4} \int d^2\sigma (\partial_1 P \partial_- P - \partial_1 Q \partial_+ Q). \quad (12)$$

We re-expand this in the non-chiral basis to give Tseytlin's [5] duality-symmetric formulation of the action

$$S = \frac{1}{2} \int d^2\sigma \left[-(R\partial_1 X)^2 - (R^{-1}\partial_1 \tilde{X})^2 + 2\partial_0 X \partial_1 \tilde{X} \right]. \quad (13)$$

Notice that the unusual normalisation of (1) was exactly what was needed for this form of the action to have the correct normalisation. The constraints

$$\partial_0 \tilde{X} = R^2 \partial_1 X \quad \partial_0 X = R^{-2} \partial_1 \tilde{X} \quad (14)$$

then follow after integrating the equations of motion⁶ and the string wave equation is given by combining the constraint equations.

⁶We fix the arbitrary function of τ introduced by integration by observing that (13) has $\delta X = f(\tau)$ gauge invariance.

Returning to the general case, the PST procedure yields an action

$$S = \frac{1}{2} \int d^2\sigma \left[-\mathcal{G}_{\alpha\beta} \partial_1 X^\alpha \partial_1 X^\beta + \mathcal{L}_{\alpha\beta} \partial_1 X^\alpha \partial_0 X^\beta + \mathcal{K}_{\alpha\beta} \partial_0 X^\alpha \partial_0 X^\beta \right], \quad (15)$$

where $X^\alpha = (\mathbb{X}^A, Y^a) = (X^i, \tilde{X}_j, Y^a)$ and⁷

$$\mathcal{G} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & g \end{pmatrix}, \mathcal{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}. \quad (16)$$

For the fibre coordinates we have the equation of motion

$$\partial_1 (\mathcal{H} \partial_1 \mathbb{X}) = L \partial_1 \partial_0 \mathbb{X}, \quad (17)$$

which integrates to give the constraint (3). This form of the action will be our starting point for calculating the beta-functional of the theory.

3 The background-field expansion

To perturbatively study ultra-violet divergences in the doubled formalism we expand quantum fluctuations around a classical background. We make a choice of coordinates that leaves the general coordinate invariance of the action manifest in the perturbative expansion [14, 15].

First one writes the fields X^α as the sum of a classical piece X_{cl}^α and a quantum fluctuation π^α , however, the expansion in this fluctuation would not be general covariant. Instead one takes the geodesic from X_{cl}^α to $X_{cl}^\alpha + \pi^\alpha$ and finds its tangent vector at X_{cl}^α with length equal to that of the geodesic. We call this tangent vector ξ^α and it is contravariant, so the coefficients in an expansion in terms of ξ^α are tensors. In general, terms linear in ξ^α are proportional to the equations of motion of the classical background and so vanish. The quadratic terms will give rise to the one loop corrections and so are the relevant terms for calculating the one-loop beta-function.

We will use the algorithmic method of calculating the background-field expansion developed in [16]. To obtain the n th order background-field expanded action we

⁷In our notation we reserve the Greek characters μ and ν to denote worldsheet indices.

simply act on the Lagrangian with the operator

$$\int d^2\sigma \xi^\alpha(\underline{\sigma}) D_\alpha^\sigma \quad (18)$$

n times and divide by $n!$ (D_α^σ is the covariant functional derivative with respect to $X^\alpha(\underline{\sigma})$). The action of this operator can be summarised as

$$\int d^2\sigma \xi^\alpha(\underline{\sigma}) D_\alpha^\sigma \xi^\beta(\underline{\sigma}') = 0, \quad (19)$$

$$\int d^2\sigma \xi^\alpha(\underline{\sigma}) D_\alpha^\sigma \partial_\mu X^\beta(\underline{\sigma}') = D_\mu \xi^\beta(\underline{\sigma}'), \quad (20)$$

$$\int d^2\sigma \xi^\alpha(\underline{\sigma}) D_\alpha^\sigma D_\mu \xi^\beta(\underline{\sigma}') = R^\beta_{\alpha\gamma\delta} \partial_\mu X^\delta \xi^\alpha \xi^\gamma(\underline{\sigma}'), \quad (21)$$

$$\int d^2\sigma \xi^\alpha(\underline{\sigma}) D_\alpha^\sigma T_{\alpha_1\alpha_2\dots\alpha_n}(X(\underline{\sigma}')) = D_\beta T_{\alpha_1\alpha_2\dots\alpha_n} \xi^\beta(\underline{\sigma}'), \quad (22)$$

where $R^\beta_{\alpha\gamma\delta}$ is the target space Riemann tensor and $T_{\alpha_1\alpha_2\dots\alpha_n}$ is a rank n tensor and these are understood to be evaluated at the classical value X_{cl} . The form of (22) is particularly relevant, leading to simplification when dealing with the metric.

Expanding the first term in (15) is exactly the same as the standard sigma model calculation (albeit without world sheet covariance), at first order we have⁸

$$-\mathcal{G}_{\alpha\beta} \partial_1 X^\alpha D_1 \xi^\beta \quad (23)$$

and at second order

$$-\frac{1}{2} (\mathcal{G}_{\alpha\beta} D_1 \xi^\alpha D_1 \xi^\beta + R_{\gamma\alpha\beta\delta} \xi^\alpha \xi^\beta \partial_1 X^\gamma \partial_1 X^\delta), \quad (24)$$

where $R_{\gamma\alpha\beta\delta}$ is the Riemann tensor constructed from the metric \mathcal{G} . The expansion of the \mathcal{L} term in the action is more complex giving

$$\frac{1}{2} (\mathcal{L}_{\alpha\beta} \partial_0 X^\alpha D_1 \xi^\beta + \mathcal{L}_{\alpha\beta} D_0 \xi^\alpha \partial_1 X^\beta + D_\alpha \mathcal{L}_{\gamma\beta} \xi^\alpha \partial_0 X^\gamma \partial_1 X^\beta) \quad (25)$$

at first order and

$$\begin{aligned} \frac{1}{2} \left(\mathcal{L}_{\alpha\beta} D_0 \xi^\alpha D_1 \xi^\beta + \frac{1}{2} (D_\alpha D_\beta \mathcal{L}_{\gamma\delta} + L_{\gamma\sigma} R^\sigma_{\alpha\beta\delta} + L_{\delta\sigma} R^\sigma_{\alpha\beta\gamma}) \xi^\alpha \xi^\beta \partial_0 X^\gamma \partial_1 X^\delta \right. \\ \left. + D_\gamma \mathcal{L}_{\alpha\beta} \xi^\gamma (\partial_0 X^\alpha D_1 \xi^\beta + D_0 \xi^\alpha \partial_1 X^\beta) \right) \quad (26) \end{aligned}$$

⁸From now on X refers to the classical field X_{cl} .

at second-order. This is the general expansion for any second rank tensor \mathcal{L} so the \mathcal{K} term in (15) may be expanded in a similar way.

The first-order terms in ξ vanish as they should (using the equations of motion of X_{cl}) leaving the second-order Lagrangian which is given by

$$\begin{aligned}
2\mathcal{L}_{(2)} = & -\mathcal{G}_{\alpha\beta}D_1\xi^\alpha D_1\xi^\beta + \mathcal{L}_{\alpha\beta}D_0\xi^\alpha D_1\xi^\beta + \mathcal{K}_{\alpha\beta}D_0\xi^\alpha D_0\xi^\beta \\
& -R_{\gamma\alpha\beta\delta}\xi^\alpha\xi^\beta\partial_1X^\gamma\partial_1X^\delta + \mathcal{L}_{\alpha\beta;\gamma}\xi^\gamma(D_0\xi^\alpha\partial_1X^\beta + \partial_0X^\alpha D_1\xi^\beta) \\
& +\frac{1}{2}D_\alpha D_\beta\mathcal{L}_{\gamma\delta}\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_1X^\delta + \frac{1}{2}(L_{\gamma\sigma}R^\sigma_{\alpha\beta\delta} + L_{\delta\sigma}R^\sigma_{\alpha\beta\gamma})\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_1X^\delta \\
& +2\mathcal{K}_{\alpha\beta;\gamma}\xi^\gamma D_0\xi^\alpha\partial_0X^\beta \\
& +\frac{1}{2}D_\alpha D_\beta\mathcal{K}_{\gamma\delta}\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_0X^\delta + \mathcal{K}_{\gamma\sigma}R^\sigma_{\alpha\beta\delta}\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_0X^\delta.
\end{aligned} \tag{27}$$

4 Simplification strategy

Now we have the background-field expanded action we may simplify it using the equations of motion of X^α (recall X^α is the classical field configuration which we are expanding around and so obeys its equation of motion). Then we use Wick contractions on the quantum field ξ . The equation of motion is

$$D_1(\mathcal{G}_{\alpha\beta}\partial_1X^\beta) = \mathcal{L}_{\alpha\beta}\partial_1\partial_0X^\beta + \hat{D}_0(\mathcal{K}_{\alpha\beta}\partial_0X^\beta), \tag{28}$$

where \hat{D}_0 is a covariant derivative constructed only from the base metric g ($\hat{D}_0\mathcal{K} = 0$ and $\hat{\Gamma}$ will similarly refer to the connection constructed from g). This equation is required to show vanishing of the first-order action and we now use it to remove all \mathcal{L} terms (except the \mathcal{L} fluctuation ‘kinetic’ term) from the action. This means (using integration by parts) we can substitute

$$\begin{aligned}
& \mathcal{L}_{\alpha\beta;\gamma}\xi^\gamma(\partial_0\xi^\alpha\partial_1X^\beta + \partial_1\xi^\alpha\partial_0X^\beta) + \mathcal{L}_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\xi^\gamma\partial_1X^\delta\partial_0\xi^\beta + \mathcal{L}_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}\xi^\gamma\partial_0X^\delta\partial_1\xi^\beta \\
= & \frac{1}{2}\mathcal{L}_{\beta\delta}\partial_\sigma\Gamma^\delta_{\gamma\alpha}\xi^\gamma\xi^\alpha(\partial_1X^\beta\partial_0X^\sigma + \partial_0X^\beta\partial_1X^\sigma) \\
& -2\mathcal{G}_{\delta\beta}\Gamma^\delta_{\gamma\alpha}\xi^\gamma\partial_1\xi^\alpha\partial_1X^\beta - \partial_\sigma(\mathcal{G}_{\delta\beta}\Gamma^\delta_{\gamma\alpha})\xi^\gamma\xi^\alpha\partial_1X^\beta\partial_1X^\sigma \\
& +2\mathcal{K}_{\delta\beta}\Gamma^\delta_{\gamma\alpha}\xi^\gamma\partial_0\xi^\alpha\partial_0X^\beta + \partial_\sigma(\mathcal{K}_{\delta\beta}\Gamma^\delta_{\gamma\alpha}\xi^\gamma)\xi^\alpha\partial_0X^\beta\partial_0X^\sigma \\
& +\Gamma^\delta_{\gamma\alpha}\xi^\gamma\xi^\alpha\left(\mathcal{G}_{\delta\beta}\Gamma^\beta_{\sigma\tau}\partial_1X^\sigma\partial_1X^\tau - \mathcal{K}_{\delta\beta}\hat{\Gamma}^\beta_{\sigma\tau}\partial_0X^\sigma\partial_0X^\tau\right).
\end{aligned} \tag{29}$$

This leads to a dramatically simplified Lagrangian given by

$$\begin{aligned}
2\mathcal{L}_{(2)} = & -\mathcal{G}_{\alpha\beta}\partial_1\xi^\alpha\partial_1\xi^\beta + \mathcal{L}_{\alpha\beta}\partial_0\xi^\alpha\partial_1\xi^\beta + \mathcal{K}_{\alpha\beta}\partial_0\xi^\alpha\partial_0\xi^\beta \\
& -2\partial_\alpha\mathcal{G}_{\gamma\beta}\partial_1X^\gamma\xi^\alpha\partial_1\xi^\beta - \frac{1}{2}\partial_\alpha\partial_\beta\mathcal{G}_{\gamma\delta}\partial_1X^\gamma\partial_1X^\delta\xi^\alpha\xi^\beta \\
& +2\partial_\alpha g_{\beta\gamma}\xi^\alpha\partial_0\xi^\beta\partial_0X^\gamma + \frac{1}{2}\partial_\alpha\partial_\beta\mathcal{K}_{\gamma\delta}\xi^\alpha\xi^\beta\partial_0X^\gamma\partial_0X^\delta .
\end{aligned} \tag{30}$$

Note that we have chosen to proceed by expanding covariant derivatives and simplifying using the equations of motion, rather than leaving things expressed in terms of covariant derivatives. We now proceed to introduce vielbeins so that we can work in the chiral frame where we know how to find the fluctuation propagators.

Once vielbeins are introduced there are of course terms with derivatives acting on the vielbeins. Normally such terms are accounted for by exchanging the usual connection for the spin connection. The pull back of the spin connection to the world sheet transforms as a gauge field. There is then a general argument that this gauge field, which is minimally coupled, cannot contribute to the Weyl anomaly. We have a modified action where the gauge connection is no longer be minimally coupled and there is no such argument (indeed we find contributions from the ‘gauge’ terms).

Introducing the vielbein has the effect of moving all indices on ξ ’s in the second-order action to the chiral frame at the expense of also introducing the following terms:

$$\begin{aligned}
2\mathcal{L}_\mathcal{V} = & -2\mathcal{G}_{\bar{\alpha}\beta}\partial_1\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\beta}}\partial_1\xi^{\bar{\alpha}} - \mathcal{G}_{\alpha\beta}\partial_1\mathcal{V}_{\bar{\beta}}^\beta\partial_1\mathcal{V}_{\bar{\alpha}}^\alpha\xi^{\bar{\beta}}\xi^{\bar{\alpha}} \\
& +\mathcal{L}_{\bar{\alpha}\beta}\partial_1\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\beta}}\partial_0\xi^{\bar{\alpha}} + \mathcal{L}_{\bar{\alpha}\beta}\partial_0\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\beta}}\partial_1\xi^{\bar{\alpha}} + \mathcal{L}_{\alpha\beta}\partial_1\mathcal{V}_{\bar{\beta}}^\beta\partial_0\mathcal{V}_{\bar{\alpha}}^\alpha\xi^{\bar{\beta}}\xi^{\bar{\alpha}} \\
& +2\mathcal{K}_{\bar{\alpha}\beta}\partial_0\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\beta}}\partial_0\xi^{\bar{\alpha}} + \mathcal{K}_{\alpha\beta}\partial_0\mathcal{V}_{\bar{\beta}}^\beta\partial_0\mathcal{V}_{\bar{\alpha}}^\alpha\xi^{\bar{\beta}}\xi^{\bar{\alpha}} \\
& -2\partial_{\bar{\alpha}}\mathcal{G}_{\gamma\beta}\partial_1X^\gamma\partial_1\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\alpha}}\xi^{\bar{\beta}} + 2\partial_{\bar{\alpha}}\mathcal{K}_{\gamma\beta}\partial_0X^\gamma\partial_0\mathcal{V}_{\bar{\beta}}^\beta\xi^{\bar{\alpha}}\xi^{\bar{\beta}} .
\end{aligned} \tag{31}$$

5 Wick contraction

In this chiral frame \mathcal{L} and \mathcal{H} are diagonal and one can calculate the propagators for the fluctuations from the ‘kinetic terms’ in the Lagrangian. On the fibre these ‘kinetic terms’ are FJ style Lagrangians of the form (12) for n chiral and n anti-chiral Bosons in flat space. The propagators for such chiral Lagrangians have previously

been considered by Tseytlin [5]. The sum of a chiral and anti-chiral propagator is proportional to a standard Boson propagator Δ_0 . The difference of chiral and anti-chiral propagators gives a phase θ . Full details of this are given in the appendix. The general result for our action is that

$$\langle \xi^{\bar{A}} \xi^{\bar{B}} \rangle = \Delta_0 \mathcal{H}^{\bar{A}\bar{B}} + \theta L^{\bar{A}\bar{B}}. \quad (32)$$

Δ_0 contains UV divergence that needs regularisation and renormalisation. The coefficients of Δ_0 will thus contribute to the Weyl anomaly and in turn, to the beta-functionals.

θ does not contain any divergence and does not therefore contribute to the Weyl anomaly. Instead θ parameterises any Lorentz anomaly. One would demand such anomaly vanishes by setting $\frac{\partial S_{eff}}{\partial \theta} = 0$. This would place additional constraints on the background fields beyond that of the beta-functionals vanishing. However, when the dust settles all occurrences of θ in the effective action cancel out leaving no Lorentz anomaly. This is as expected since we have an equal number of Bosons of each chirality.

Given (32) we can deduce the form of more complicated Wick contractions which are quartic in ξ and contain derivatives of fluctuations⁹. These contractions arise in the $O(S_{eff}^2)$ term in the exponential of the effective action and must be included since we need to count all logarithmic divergences (see for example [15]).

When evaluating all contributing terms it will be useful to distinguish ‘base terms’ which contain only the base metric g and its derivatives and ‘vielbein terms’ which come from (31) and contain derivatives of the vielbein prior to any integrations by parts.

5.1 Single contraction terms

The terms with a single $\langle \xi \xi \rangle$ contraction are

$$\frac{1}{2} \partial_a \partial_b g_{bg} \langle \xi^a \xi^b \rangle \partial_\mu Y^g \partial^\mu Y^d = \frac{1}{2} \partial^a \partial_a g_{gd} \partial_\mu Y^g \partial^\mu Y^d \Delta_0 \quad (33)$$

⁹See appendix.

from the base,

$$-\frac{1}{2}\partial_a\partial_b\mathcal{H}_{GD}\partial_1\mathbb{X}^G\partial_1\mathbb{X}^D\langle\xi^a\xi^b\rangle = -\frac{1}{2}\partial_a\partial^a\mathcal{H}_{GD}\partial_1\mathbb{X}^G\partial_1\mathbb{X}^D\Delta_0 \quad (34)$$

on the fibre and

$$-\mathcal{H}_{AB}\partial_1\mathcal{V}_{\bar{B}}^B\partial_1\mathcal{V}_{\bar{A}}^A\langle\xi^{\bar{B}}\xi^{\bar{A}}\rangle = \left(-\partial_1\mathcal{V}_{\bar{A}}^A\partial_1\mathcal{V}_{\bar{A}}^{\bar{A}} + \frac{1}{2}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB}\right)\Delta_0, \quad (35)$$

$$L_{AB}\partial_1\mathcal{V}_{\bar{B}}^B\partial_0\mathcal{V}_{\bar{A}}^A\langle\xi^{\bar{B}}\xi^{\bar{A}}\rangle = \left(\partial_1\mathcal{V}_{\bar{A}}^{\bar{A}}\partial_0\mathcal{V}_{\bar{B}}^A\delta^{\bar{B}\bar{C}}L_{\bar{C}\bar{A}}\right)\Delta_0, \quad (36)$$

$$-2\partial_{\bar{a}}\mathcal{H}_{BG}\partial_1\mathbb{X}^G\partial_1\mathcal{V}_{\bar{B}}^B\langle\xi^{\bar{a}}\xi^{\bar{B}}\rangle = 0, \quad (37)$$

$$g_{ab}\partial^\mu\mathcal{V}_{\bar{b}}^b\partial_\mu\mathcal{V}_{\bar{a}}^a\langle\xi^{\bar{b}}\xi^{\bar{a}}\rangle = \left(\partial^\mu\mathcal{V}_{\bar{a}}^a\partial_\mu\mathcal{V}_{\bar{a}}^{\bar{a}} - \frac{1}{2}\partial^\mu g_{ab}\partial_\mu g^{ab}\right)\Delta_0, \quad (38)$$

$$2\partial_{\bar{a}}g_{bg}\partial^\mu Y^g\partial_\mu\mathcal{V}_{\bar{b}}^b\langle\xi^{\bar{a}}\xi^{\bar{b}}\rangle = \left(-2\partial_a g_{gb}\partial_d\mathcal{V}_{\bar{b}}^b\delta^{\bar{a}\bar{b}}\mathcal{V}_{\bar{a}}^a\right)\Delta_0 \quad (39)$$

from the vielbeins.

5.2 Double contraction terms

These occur when expanding the exponential of the effective action to second order. Although there seem myriad possible terms that could contribute, especially from vielbein terms, many vanish trivially. This because new divergent diagrams must still be one-loop in fluctuations so contain one ‘loop’ of indices: the block diagonal form of the metrics and vielbeins ensure the terms mainly separate into base and fibre terms, with a few ‘cross-terms’. We use the propagator contractions described in Appendix A and note that these terms are a factor of a half down due the exponential, and a further factor of a half down due to the two sitting on the left-hand side of (30).

On the base we get

$$\begin{aligned} & -\partial_{\bar{a}}g_{g\bar{b}}\partial^\mu Y^g\partial_{\bar{c}}g_{d\bar{e}}\partial_\mu Y^d\langle\xi^{\bar{a}}\partial_1\xi^{\bar{b}}\xi^{\bar{c}}\partial_1\xi^{\bar{e}}\rangle \\ & = -\frac{1}{2}\left(\partial_a g_{gb}g^{bc}\partial^a g_{cd} - \partial^a g_{bg}\partial^b g_{ad}\right)\partial^\mu Y^g\partial_\mu Y^d\Delta_0 \end{aligned} \quad (40)$$

and on the fibre

$$\begin{aligned} & \partial_{\bar{a}}\mathcal{H}_{GB}\partial_1\mathbb{X}^G\partial_{\bar{c}}\mathcal{H}_{D\bar{E}}\partial_1\mathbb{X}^D\langle\xi^{\bar{a}}\partial_1\xi^{\bar{B}}\xi^{\bar{c}}\partial_1\xi^{\bar{E}}\rangle \\ & = \frac{1}{2}\partial_a\mathcal{H}_{GB}\mathcal{H}^{BC}\partial^a\mathcal{H}_{CD}\partial_1\mathbb{X}^G\partial_1\mathbb{X}^D\Delta_0. \end{aligned} \quad (41)$$

Purely from the vielbein piece of the Lagrangian (31) we have

$$\begin{aligned}
\frac{1}{4}L_{\bar{A}B}\partial_1\mathcal{V}_{\bar{B}}^BL_{\bar{C}D}\partial_1\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_0\xi^{\bar{B}}\xi^{\bar{C}}\partial_0\xi^{\bar{D}}\right\rangle &= \left(-\partial_1\mathcal{V}_{\bar{A}}^A\partial_1\mathcal{V}_{\bar{A}}^{\bar{A}}+\frac{1}{8}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB}\right)\Delta_0, \\
\frac{1}{4}L_{\bar{A}B}\partial_0\mathcal{V}_{\bar{B}}^BL_{\bar{C}D}\partial_0\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_1\xi^{\bar{D}}\right\rangle &= \left(-\frac{1}{8}\partial_0\mathcal{H}_{AB}\partial_0\mathcal{H}^{AB}\right)\Delta_0, \\
\frac{1}{2}L_{\bar{A}B}\partial_0\mathcal{V}_{\bar{B}}^BL_{\bar{C}D}\partial_1\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_0\xi^{\bar{D}}\right\rangle &= \left(-\partial_1\mathcal{V}_{\bar{A}}^A\partial_0\mathcal{V}_{\bar{B}}^{\bar{B}}\hat{\delta}_{\bar{A}}^{\bar{B}}\right)\Delta_0, \\
-\mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}_{\bar{B}}^BL_{\bar{C}D}\partial_1\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_0\xi^{\bar{D}}\right\rangle &= \left(2\partial_1\mathcal{V}_{\bar{A}}^A\partial_1\mathcal{V}_{\bar{A}}^{\bar{A}}-\frac{1}{2}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB}\right)\Delta_0, \\
-\mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}_{\bar{B}}^BL_{\bar{C}D}\partial_0\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_1\xi^{\bar{D}}\right\rangle &= 0, \\
\mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}_{\bar{B}}^B\mathcal{H}_{\bar{C}D}\partial_1\mathcal{V}_{\bar{D}}^D\left\langle\xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_1\xi^{\bar{D}}\right\rangle &= 0, \\
\sum_{\mu=0,1}g_{ab}\partial^\mu\mathcal{V}_{\bar{b}}^b\partial_{\bar{c}d}\partial^\mu\mathcal{V}_{\bar{d}}^d\left\langle\xi^{\bar{a}}\partial_\mu\xi^{\bar{b}}\xi^{\bar{c}}\partial_\mu\xi^{\bar{d}}\right\rangle &= \left(\frac{1}{4}\partial^\mu g_{ab}\partial_\mu g^{ab}-\partial^\mu\mathcal{V}_{\bar{a}}^a\partial_\mu\mathcal{V}_{\bar{a}}^{\bar{a}}\right)\Delta_0, \quad (42)
\end{aligned}$$

with one cross-term

$$\begin{aligned}
&\sum_{\mu=0,1}2\partial_ag_{gb}\mathcal{V}_{\bar{b}}^b\mathcal{V}_{\bar{a}}^a\partial^\mu Y^g g_{\bar{c}e}\partial^\mu\mathcal{V}_{\bar{d}}^e\left\langle\xi^{\bar{a}}\partial_\mu\xi^{\bar{b}}\xi^{\bar{c}}\partial_\mu\xi^{\bar{d}}\right\rangle \\
&= \left(-2\partial_ag_{gb}\partial_d\mathcal{V}_{\bar{b}}^b\delta^{\bar{a}\bar{b}}\mathcal{V}_{\bar{a}}^{\bar{a}}+\partial_ag_{gb}\partial_d g^{ba}\right)\partial^\mu Y^g\partial_\mu Y^d\Delta_0 \quad (43)
\end{aligned}$$

5.3 The Weyl divergence

The total Weyl divergence will be given by the coefficient of Δ_0 which we denote by W so that

$$S_{Weyl} = \frac{1}{2}\int d^2\sigma[-W_{GD}\partial_1\mathbb{X}^G\partial_1\mathbb{X}^D+W_{gd}\partial_\mu Y^g\partial^\mu Y^d]\Delta_0. \quad (44)$$

On the base the divergence W_{gd} is given by

$$\begin{aligned}
W_{gd} &= \frac{1}{2}\partial^a\partial_ag_{gd}-\frac{1}{4}\partial_g g_{ab}\partial_d g^{ab}-\frac{1}{2}\partial_ag_{gb}g^{bc}\partial^a g_{cd} \\
&\quad +\frac{1}{2}\partial^a g_{bg}\partial^b g_{ad}+\partial_ag_{gb}\partial_d g^{ba} \\
&\quad -\frac{1}{8}\partial_g\mathcal{H}_{AB}\partial_d\mathcal{H}^{AB}. \quad (45)
\end{aligned}$$

The divergence on the fibre is

$$W_{GD} = \frac{1}{2} \partial^2 \mathcal{H}_{GD} - \frac{1}{2} ((\partial_a \mathcal{H}) \mathcal{H}^{-1} (\partial^a \mathcal{H}))_{GD}. \quad (46)$$

The divergence on the base, (45), can be rewritten as

$$\begin{aligned} W_{gd} &= g^{ab} g_{gs} (\partial_b \Gamma_{ad}^s + \Gamma_{bt}^s \Gamma_{ad}^t) \\ &\quad - \frac{1}{2} \partial_g g_{ab} \partial_d g^{ab} + \partial_a g_{gb} \partial_d g^{ba} \\ &\quad - \frac{1}{8} \partial_g \mathcal{H}_{AB} \partial_d \mathcal{H}^{AB}, \end{aligned} \quad (47)$$

where we recognise the first two terms as part of the Ricci tensor. We now, using the base components of the equation of motion for the fields, add zero to the divergence in the form

$$\Gamma_{ab}^t g^{ab} \left(\hat{D}_\mu (g_{td} \partial^\mu Y^d) - \frac{1}{2} \partial_t \mathcal{H}_{GD} \partial_1 \mathbb{X}^G \partial_1 \mathbb{X}^D \right). \quad (48)$$

The base divergence becomes

$$W_{gd} = -\hat{R}_{gd} - \frac{1}{8} \partial_g \mathcal{H}_{AB} \partial_d \mathcal{H}^{AB}, \quad (49)$$

where \hat{R}_{gd} is the Ricci tensor constructed from the base metric g alone. The fibre components of the divergence become

$$W_{GD} = \frac{1}{2} \partial^2 \mathcal{H}_{GD} - \frac{1}{2} ((\partial_a \mathcal{H}) \mathcal{H}^{-1} (\partial^a \mathcal{H}))_{GD} - \frac{1}{2} \Gamma_{ab}^t g^{ab} \partial_t \mathcal{H}_{GD} \quad (50)$$

5.4 Relation to the doubled Ricci tensor

If we calculate the Ricci tensor of the doubled space (the Ricci tensor of \mathcal{G}) for comparison, and drop terms proportional to $\mathcal{H}^{AB} \partial_d \mathcal{H}_{AB} = 0$, it also has block diagonal form with

$$R_{GD} = -\frac{1}{2} \partial^2 \mathcal{H}_{GD} + \frac{1}{2} ((\partial_a \mathcal{H}) \mathcal{H}^{-1} (\partial^a \mathcal{H}))_{GD} + \frac{1}{2} \Gamma_{ab}^t g^{ab} \partial_t \mathcal{H}_{GD}, \quad (51)$$

$$R_{gd} = \hat{R}_{ab} + \frac{1}{4} \text{tr}(\partial_g \mathcal{H} \partial_d \mathcal{H}^{-1}), \quad (52)$$

for the fibre and base parts respectively. We see that the Weyl divergence are almost equal to minus the Ricci tensor except that the term on the base containing the doubled metric \mathcal{H} have an extra factor of $1/2$. We note also that the fibre divergence is contracted with $\partial_1 \mathbb{X} \partial_1 \mathbb{X}$, whereas if we considered an ordinary sigma model with metric \mathcal{G} the fibre piece would be contracted with $\partial^\mu \mathbb{X} \partial_\mu \mathbb{X}$. However, we can use the fibre equations of motion to make W_{GD} contract $\partial^\mu \mathbb{X} \partial_\mu \mathbb{X}$ at the expense of introducing a factor of $1/2$. Then, comparing W with R , all terms containing the doubled metric \mathcal{H} would be a factor of $1/2$ down. We will see that writing the doubled metric in terms of the standard sigma model fields h and b takes care of these extra factors.

6 Doubled renormalisation

One may now proceed directly to regularise and renormalise the divergences coming from Δ_0 . In the standard way one would dimensionally regularise and introduce a mass scale μ through, say, minimal subtraction and the introduction of counter terms. We then absorb all scale dependence to define the renormalised couplings

$$\{\mathcal{G}, \mathcal{K}, \mathcal{L}\} \rightarrow \{\mathcal{G}^R(\mu), \mathcal{K}^R(\mu), \mathcal{L}^R(\mu)\} \quad (53)$$

producing the renormalised action

$$S^R = \frac{1}{2} \int d^2 \sigma \left[-\mathcal{G}^R(\mu)_{\alpha\beta} \partial_1 X^\alpha \partial_1 X^\beta + \mathcal{L}^R(\mu)_{\alpha\beta} \partial_1 X^\alpha \partial_0 X^\beta + \mathcal{K}^R(\mu)_{\alpha\beta} \partial_0 X^\alpha \partial_0 X^\beta \right]. \quad (54)$$

We can calculate the beta-functions from this by differentiating the renormalised couplings with respect to the log of the mass scale giving

$$\beta_{\alpha\beta}^{\mathcal{G}} = - \begin{pmatrix} W_{AB} & 0 \\ 0 & W_{ab} \end{pmatrix}, \quad \beta_{\alpha\beta}^{\mathcal{K}} = - \begin{pmatrix} 0 & 0 \\ 0 & W_{ab} \end{pmatrix}, \quad \beta_{\alpha\beta}^{\mathcal{L}} = 0. \quad (55)$$

Demanding the vanishing of these beta-functions gives the background field equations.

7 Equivalence with standard sigma model

Instead of working directly with these doubled beta-functions we shall show the equivalence to the standard sigma model by expanding out the Weyl divergence in terms

of the non-doubled metric and B-field and eliminate the extra doubled coordinates before renormalisation. Since this can be cast in a well known form for trivial base metric we will proceed putting $g_{ab} = \delta_{ab}$. Expanding the Weyl divergence $W_{\alpha\beta}$ using (2) we obtain

$$W_{AB} = \frac{1}{2} \begin{pmatrix} (r + bh^{-1}rh^{-1}b - bh^{-1}s - sh^{-1}b)_{ij} & (sh^{-1} - bh^{-1}rh^{-1})_i{}^j \\ - (h^{-1}s - h^{-1}rbh^{-1})^i{}_j & - (h^{-1}rh^{-1})^{ij} \end{pmatrix}, \quad (56)$$

$$W_{ab} = \frac{1}{4}t_{ab}, \quad (57)$$

where we have defined

$$r_{ij} = (\partial^2 h - \partial_a h h^{-1} \partial^a h - \partial_a b h^{-1} \partial^a b)_{ij}, \quad (58)$$

$$s_{ij} = (\partial^2 b - \partial_a b h^{-1} \partial^a h - \partial_a h h^{-1} \partial^a b)_{ij}, \quad (59)$$

$$t_{ab} = \text{tr} (h^{-1} \partial_a h h^{-1} \partial_b h - h^{-1} \partial_a b h^{-1} \partial_b b). \quad (60)$$

Recall that $X^A = \mathbb{X}^A = (X^i, \tilde{X}_i)$. We now wish to eliminate the dual coordinates \tilde{X}_i from the Weyl divergence using the constraint

$$d\mathbb{X}^A = L^{AB} \mathcal{H}_{BC} * d\mathbb{X}^C, \quad (61)$$

which implies that

$$\partial_1 \tilde{X}_i = h_{ij} \partial_0 X^j + b_{ij} \partial_1 X^j. \quad (62)$$

We can observe that the right hand side of the above has a sensible interpretation in terms of the standard sigma model; it is proportional to the canonical momentum. On using the constraint we find that

$$W_{AB} \partial_1 \mathbb{X}^A \partial_1 \mathbb{X}^B = \frac{1}{2} r_{ij} \partial_\mu X^i \partial^\mu X^j + \frac{1}{2} \epsilon^{\mu\nu} s_{ij} \partial_\mu X^i \partial_\nu X^j. \quad (63)$$

Thus, we find that prior to renormalisation, the Weyl divergence part of the effective action is

$$S_{Weyl} = \frac{1}{2} \int d^2 \sigma [-W_{AB} \partial_1 \mathbb{X}^A \partial_1 \mathbb{X}^B + W_{ab} \partial_\mu Y^a \partial^\mu Y^b] \Delta_0 \quad (64)$$

$$= \frac{1}{4} \int d^2 \sigma \left[r_{ij} \partial_\mu X^i \partial^\mu X^j + s_{ij} \epsilon^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + \frac{t_{ab}}{2} \partial_\mu Y^a \partial^\mu Y^b \right] \Delta_0. \quad (65)$$

Demanding that this divergence vanishes constrains the background fields to obey $r = s = t = 0$.

We now wish to compare this to the standard sigma model in conformal gauge

$$S = \frac{1}{2} \int d^2\sigma \left[G_{\alpha\beta} \partial_\mu X^\alpha \partial^\mu X^\beta + \epsilon^{\mu\nu} B_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta \right] \quad (66)$$

with metric and B-field

$$G_{\alpha\beta} = \begin{pmatrix} h_{ij}(Y) & 0 \\ 0 & \delta_{ab} \end{pmatrix}, \quad B_{\alpha\beta} = \begin{pmatrix} b_{ij}(Y) & 0 \\ 0 & 0 \end{pmatrix}. \quad (67)$$

The beta-functionals for this sigma model are [17]¹⁰

$$\beta_{\alpha\beta}^G = R_{\alpha\beta} - \frac{1}{4} H_{\alpha\sigma\delta} H_\beta^{\sigma\delta}, \quad (68)$$

$$\beta_{\alpha\beta}^B = -\frac{1}{2} D^\sigma H_{\sigma\alpha\beta}, \quad (69)$$

where $H_{\alpha\beta\sigma} = \partial_\alpha B_{\beta\sigma} + \partial_\beta B_{\sigma\alpha} + \partial_\sigma B_{\alpha\beta}$. On substitution of our ansatz for B and G we find that the non-vanishing components are

$$\beta_{ij}^G = -\frac{1}{2} \left(r + \frac{1}{2} \text{tr} (h^{-1} \partial_a h) \partial^a h \right)_{ij}, \quad (70)$$

$$\beta_{ab}^G = -\frac{1}{2} \left(\frac{t_{ab}}{2} + \partial_a \text{tr} (h^{-1} \partial_b h) \right), \quad (71)$$

$$\beta_{ij}^B = -\frac{1}{2} \left(s + \frac{1}{2} \text{tr} (h^{-1} \partial_a h) \partial^a b \right)_{ij}, \quad (72)$$

$$\beta_{ab}^B = \frac{1}{4} \text{tr} (h^{-1} \partial_a b h^{-1} \partial_b h + h^{-1} \partial_a h h^{-1} \partial_b b). \quad (73)$$

The Weyl divergent part of the effective action which produces these beta-functions after renormalisation is,

$$S_{Weyl} = -\frac{1}{2} \int d^2\sigma \left[\beta_{ij}^G \partial_\mu X^i \partial^\mu X^j + \beta_{ab}^G \partial_\mu Y^a \partial^\mu Y^b + \epsilon^{\mu\nu} (\beta_{ij}^B \partial_\mu X^i \partial_\nu X^j + \beta_{ab}^B \partial_\mu Y^a \partial_\nu Y^b) \right] \Delta_0. \quad (74)$$

¹⁰We have set the dilaton to a constant.

The anti-symmetry of $\epsilon^{\mu\nu}\partial_\mu Y^a\partial_\nu Y^b$ allows us to cancel the divergence that gave rise to β_{ab}^B . The equation of motion for the base coordinate Y is

$$2\partial^2 Y^a = \partial^a h_{ij}\partial_\mu X^i\partial^\mu X^j + \partial^a b_{ij}\epsilon^{\mu\nu}\partial_\mu X^i\partial_\nu X^j, \quad (75)$$

so upon multiplying both sides $\text{tr}(h^{-1}\partial_a h)$ and integrating by parts we have

$$\frac{1}{2} \int d^2\sigma \text{tr}(h^{-1}\partial_a h) (\partial^a h_{ij}\delta^{\mu\nu} + \partial^a b_{ij}\epsilon^{\mu\nu}) \partial_\mu X^i\partial_\nu X^j \quad (76)$$

$$= \int d^2\sigma \text{tr}(h^{-1}\partial_a h) \partial^2 Y^a = - \int d^2\sigma \partial_a \text{tr}(h^{-1}\partial_b h) \partial_\mu Y^a \partial^\mu Y^b \quad (77)$$

so that (74) reduces to

$$S_{Weyl} = \frac{1}{4} \int [r_{ij}\partial_\mu X^i\partial^\mu X^j + s_{ij}\epsilon^{\mu\nu}\partial_\mu X^i\partial_\nu X^j + \frac{t_{ab}}{2}\partial_\mu Y^a\partial^\mu Y^b] \Delta_0. \quad (78)$$

This agrees with what we found previously from the doubled formalism in (65). Thus after integrating out the dual coordinate the doubled formalism gives *exactly* the same divergent terms as the standard string sigma model.

The construction can be straightforwardly extended to include a non-trivial base metric $g(Y)$. In this case the following additional terms

$$\frac{1}{2}\Gamma^t_{ab}g^{ab}\partial_t h_{ij}, \quad (79)$$

$$\hat{R}_{ab} + \frac{1}{4}\text{tr}(h^{-1}\partial^t h)\partial_t g_{ab} - \frac{1}{2}\text{tr}(h^{-1}\partial^t h)\partial_a g_{tb}, \quad (80)$$

$$\frac{1}{2}\Gamma^t_{ab}g^{ab}\partial_t b_{ij}, \quad (81)$$

are required to reproduce the usual beta-functionals (70), (71) and (72) respectively. These terms do indeed follow from the doubled geometry Weyl divergences (49) and (50) after application of the equations of motion. In fact one can immediately see \hat{R}_{ab} , the Ricci tensor of the base metric g in (49).

8 Conclusion

In summary, we have been able to calculate the one-loop Weyl divergence of the duality symmetric string which upon renormalisation gives rise to the beta-functionals for the

doubled geometry metric couplings. For the fibre coordinates these are the obvious geometric quantity, the doubled target space Ricci tensor. For base coordinates, the terms in the Ricci tensor that contain the fibre metric \mathcal{H} pick up an extra factor of a half. However, when we interpreted the results in terms of the non-doubled fields these factors are taken care of and indeed we are left with exactly the same Weyl divergence as for the standard sigma model.

In this calculation there are some notable features that we wish to draw attention to. First, the topological term which is crucial in establishing equivalence of partition functions with the standard sigma model, played no role in this calculation. Second, the non-covariant structure of the action (15) meant that unlike the calculation for the standard string, the ‘gauge’ terms do make a contribution to the divergence and increase the computational difficulty. Third, the B-field is incorporated into the doubled metric and there is no anti-symmetric term in the action. Fourth, the chiral nature of the fibre coordinates suggests that one should be concerned about any potential Lorentz anomaly. This anomaly actually vanished, cancelling between the Bosons of opposite chirality. Finally, we found that the \mathcal{L} coupling containing the $O(n, n)$ fibre metric L does not get renormalised.

8.1 Discussion

There are a number of assumptions in this work that would be interesting to explore further. We assumed that the connection in the fibration was identically zero. To include such a connection would add off diagonal elements to the doubled-space metric and would also require a suitably generalised constraint (with derivatives promoted to covariant derivatives). On applying the PST procedure to produce an action akin to (15) one finds that terms involving the connection appear in both the metric \mathcal{G} and the ‘invariant’ $O(d, d)$ metric \mathcal{L} . Evaluation of the divergence in this case would be more challenging.

As with other treatments of the duality symmetric formalism we had to specify that the fibre metric depended only on the base coordinates. It would be nice to relax this assumption. The difficulty with doing so is that the chirality constraint

would have to be modified as would its implementation with a suitable generalisation of the PST action. Another interesting and perhaps more democratic generalisation along this line would be the doubling of all coordinates. It is remarkable that the background field equations obtained required no use of the presence of any Killing directions implying that the doubled formalism is more general than one might have been first led to believe.

In this paper we have not included the doubled space dilaton, Φ , which is related to the standard dilaton, ϕ , through

$$\Phi = \phi - \frac{1}{2} \ln \det h. \quad (82)$$

This is introduced into doubled formalism with the usual Fradkin–Tseytlin action

$$S_{dil} = \frac{1}{8\pi} \int d^2\sigma \Phi(Y) R^{(2)}. \quad (83)$$

The doubled dilaton is T-dual invariant. However integrating out the dual coordinates from the doubled action (15) produces a determinant factor which correctly reproduces the transformation of the standard dilaton under T-duality [4]. Similar invariant dilatons have been used in other treatments of T-duality as review in [18].

The relation between the two dilatons (82) tells us that if we wish to set the doubled dilaton Φ to be a constant it is inconsistent to also set ϕ also to be a constant as we did when showing the equivalence of Weyl divergence. Instead, one should have $\phi = \frac{1}{2} \ln \det h$ and consider the full beta-functions for the metric and B-field in the standard sigma model

$$\beta_{\alpha\beta}^G = R_{\alpha\beta} + D_\alpha D_\beta \phi, \quad (84)$$

$$\beta_{\alpha\beta}^B = -\frac{1}{2} D^\delta H_{\alpha\beta\delta} + \frac{1}{2} D^\delta \phi H_{\alpha\beta\delta}. \quad (85)$$

The addition of these dilaton terms actually produces an exact match to the doubled space beta-functions without further manipulation using the equations of motion as we did in Section 7.

A further generalisation is to consider a general non-constant dilaton in the doubled theory producing a beta-functional β^Φ . It is clear that the leading term in β^Φ

will be proportional to $26 - c$ as is the case for the standard string. The 26 comes from an integration over world sheet metrics which remains unchanged upon doubling the target space. c is the central charge of the theory which remains the same after doubling since although we now have $d + 2n$ Bosons (d being the dimension of the non-doubled space), $2n$ of these are chiral and so contribute only a half each to the central charge. To be concrete about the full equivalence of the beta-functions with a general dilaton would require extending our analysis through to two loops. Indeed, higher loop analysis could still provide interesting corrections that are not present at leading order.

9 Acknowledgements

DSB is supported by EPSRC grant GR/R75373/02 and would like to thank DAMTP and Clare Hall college Cambridge for continued support. NBC would also like to thank DAMTP for continued support. This work was in part supported by the EC Marie Curie Research Training Network, MRTN-CT-2004-512194. DCT is supported by a STFC studentship. We would like to thank Chris Hull, James Lucietti, Andrew Low and Malcolm Perry for discussions.

A Propagators

To find the propagators for the fluctuations we look at the kinetic terms for the scalars when we have rotated the Lagrangian (30) to the chiral frame. The fluctuations have either chiral (+) or anti-chiral (-) Floreanini-Jackiw style kinetic terms with action

$$S_{\pm} = \pm \frac{1}{2} \int d^2\sigma \phi \partial_{\mp} \partial_1 \phi. \quad (86)$$

This action yields momentum-space loop propagators

$$\Delta_{\pm}(\sigma, \sigma') = \pm i \int \frac{d^2p}{(2\pi)^2} \frac{1}{p_1 p_{\mp}} e^{-ip \cdot (\sigma - \sigma')}, \quad (87)$$

and we will normally write Δ_{\pm} to indicate the $\sigma \rightarrow \sigma'$ limit. Simply by examining the integrals we see

$$\Delta_+ + \Delta_- = 2\Delta_0, \quad (88)$$

$$\Delta_+ - \Delta_- = 2\theta, \quad (89)$$

where $\Delta_0(\sigma - \sigma') = -i \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2} e^{-ip \cdot (\sigma - \sigma')}$ is the propagator for a non-chiral Boson¹¹ and we take this as the definition of θ . Of course these propagator integrals are divergent and we must regularise and then renormalise to find the beta-function.

The propagators can be calculated in z -space after Wick rotation [5] with $z = \sigma + it = \sigma + \tau$ and $\partial_\sigma = \partial + \bar{\partial}$, $\partial_\tau = \partial - \bar{\partial}$. Using a $z \rightarrow 0$ regularisation such that $\bar{\partial} z^{-1} = \pi \delta^{(2)}(z)$ one finds

$$\Delta_+(z, z') = -\frac{1}{2\pi} \ln(z - z'), \quad (90)$$

$$\Delta_-(z, z') = -\frac{1}{2\pi} \ln(\bar{z} - \bar{z}'), \quad (91)$$

$$\Delta_+(z, z') + \Delta_-(z, z') = -\frac{1}{2\pi} \ln |z - z'|^2 = 2\Delta_0(z, z'), \quad (92)$$

$$\Delta_+(z, z') - \Delta_-(z, z') = -\frac{1}{2\pi} \ln \frac{z - z'}{\bar{z} - \bar{z}'} = -\frac{i}{\pi} \arg(z - z') = 2\theta, \quad (93)$$

where in (92) we have noted after regularisation we have the same relation as (88) to the standardly normalised two-dimensional scalar propagator in this regularisation scheme.

Terms in the path integral of the effective action that are proportional to Δ_0 will be those related to a breakdown in Weyl invariance whereas terms proportional to θ will correspond to a breakdown in worldsheet Lorentz invariance [5, 6]. Looking at the form of (92) and (93) we can see scaling of z shifts Δ_0 and not θ and rotation by a phase shifts θ and not Δ_0 . One can also see that in flat space the propagator between two X coordinates or two \tilde{X} coordinates is Δ_0 whereas between an X and an \tilde{X} it is θ . The beta-function should come from terms proportional to Δ_0 in the exponential of the the action and one can obtain it by regularising and renormalising

¹¹We use $(+, -)$ signature on the worldsheet.

Δ_0 using dimensional regularisation as is standard. Terms proportional to θ would mean additional constraints on the background to ensure worldsheet Lorentz invariance and would indicate a difference from the ordinary sigma model. We find such terms cancel giving agreement with the standard formulation. We would expect this as we have equal numbers of each chirality of Boson.

Looking again at our general d -dimensional doubled action for the fibre coordinates in the chiral frame (indicated by barred indices) we have fluctuation ‘kinetic terms’ given by

$$S = \frac{1}{2} \int \left[-\mathcal{H}_{\bar{A}\bar{B}} \partial_1 \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} + L_{\bar{A}\bar{B}} \partial_1 \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \right], \quad (94)$$

where in this frame the metrics are diagonal:

$$\mathcal{H} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (95)$$

Thus the general propagator for $\xi^{\bar{A}}$ is given by

$$\langle \xi^{\bar{A}}(z) \xi^{\bar{B}}(z) \rangle = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \Delta_+ + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \Delta_- \quad (96)$$

$$= \frac{1}{2}(\mathcal{H} + L)\Delta_+ + \frac{1}{2}(\mathcal{H} - L)\Delta_- \quad (97)$$

$$= \frac{1}{2}\mathcal{H}^{\bar{A}\bar{B}}(\Delta_+ + \Delta_-) + \frac{1}{2}L^{\bar{A}\bar{B}}(\Delta_+ - \Delta_-) \quad (98)$$

$$= \Delta_0 \mathcal{H}^{\bar{A}\bar{B}} + \theta L^{\bar{A}\bar{B}}. \quad (99)$$

We can use this result and a Wick contraction procedure, described for the ordinary string in [15], to determine the divergent behaviour of higher-order propagator contractions which appear in the expansion of the exponential of our action.

For example

$$\begin{aligned}
& i \int d^2 \sigma' \left\langle \xi(\sigma)^{\bar{A}} \partial_1 \xi(\sigma)^{\bar{B}} \xi(\sigma')^{\bar{C}} \partial_0 \xi(\sigma')^{\bar{D}} \right\rangle \\
&= i \int d^2 \sigma' \left(\frac{1}{2} (\mathcal{H} + L)^{\bar{A}\bar{C}} \Delta_+(p) + \frac{1}{2} (\mathcal{H} - L)^{\bar{A}\bar{C}} \Delta_-(p) \right) e^{ip \cdot (\sigma - \sigma')} \\
&\quad \times \left(\frac{1}{2} (\mathcal{H} + L)^{\bar{B}\bar{D}} \Delta_+(q) + \frac{1}{2} (\mathcal{H} - L)^{\bar{B}\bar{D}} \Delta_-(q) \right) q_1 q_0 e^{iq \cdot (\sigma - \sigma')} \\
&\quad + i \int d^2 \sigma' \left(\frac{1}{2} (\mathcal{H} + L)^{\bar{A}\bar{D}} \Delta_+(p) + \frac{1}{2} (\mathcal{H} - L)^{\bar{A}\bar{D}} \Delta_-(p) \right) p_1 e^{ip \cdot (\sigma - \sigma')} \\
&\quad \times \left(\frac{1}{2} (\mathcal{H} + L)^{\bar{B}\bar{C}} \Delta_+(q) + \frac{1}{2} (\mathcal{H} - L)^{\bar{B}\bar{C}} \Delta_-(q) \right) q_0 e^{iq \cdot (\sigma - \sigma')}. \tag{100}
\end{aligned}$$

The σ' integral gives $(2\pi)^2 \delta(p + q)$ and putting in the forms of the integrals in Δ_{\pm} from (87) gives

$$\begin{aligned}
& i \int d^2 \sigma' \left\langle \xi(\sigma)^{\bar{A}} \partial_1 \xi(\sigma)^{\bar{B}} \xi(\sigma')^{\bar{C}} \partial_0 \xi(\sigma')^{\bar{D}} \right\rangle \\
&= \frac{i}{8} \int \frac{d^2 p}{(2\pi)^2} \left((\mathcal{H} - L) \left(\frac{1}{p_1} - \frac{1}{p_+} \right) - (\mathcal{H} + L) \left(\frac{1}{p_1} + \frac{1}{p_-} \right) \right)^{\bar{A}\bar{C}} \\
&\quad \times \left(\frac{(\mathcal{H} - L)}{p_+} - \frac{(\mathcal{H} + L)}{p_-} \right)^{\bar{B}\bar{D}} - (\bar{A} \leftrightarrow \bar{B}) \tag{101}
\end{aligned}$$

$$\begin{aligned}
&\simeq \frac{1}{8} \left(\begin{array}{c} -(\mathcal{H}\mathcal{H} - L\mathcal{H} - \mathcal{H}L + LL) \Delta_- \\ +(\mathcal{H}\mathcal{H} + L\mathcal{H} - \mathcal{H}L - LL) (-\Delta_- + \Delta_0) \\ +(\mathcal{H}\mathcal{H} - L\mathcal{H} + \mathcal{H}L - LL) (\Delta_+ - \Delta_0) \\ +(\mathcal{H}^{\bar{A}\bar{C}} \mathcal{H}^{\bar{B}\bar{D}} + L\mathcal{H} + \mathcal{H}L + LL) \Delta_+ \end{array} \right) - (\bar{A} \leftrightarrow \bar{B}) \tag{102}
\end{aligned}$$

$$= -\Delta_0 \left(\mathcal{H}^{\bar{A}\bar{C}} L^{\bar{B}\bar{D}} + L^{\bar{A}\bar{C}} \mathcal{H}^{\bar{B}\bar{D}} \right) - \theta L^{\bar{A}\bar{C}} L^{\bar{B}\bar{D}} - (\bar{A} \leftrightarrow \bar{B}), \tag{103}$$

where \simeq indicates equality up to convergent terms which are irrelevant for our purpose.

A similar procedure can be used to calculate the two-propagator contractions with any combination of worldsheet derivatives. One can also allow indices on the base; when all four indices are on the base the calculations are as for the standard sigma model (see for example [15]) and since there is no base–fibre propagator the only other allowed possibility is to have two indices on the base and two on the fibre. The results

can be compactly summarised in terms of the total space metric \mathcal{G} and \mathcal{L} as¹²

$$\langle \xi^{\bar{\alpha}} \partial_1 \xi^{\bar{\beta}} \partial_1 \xi^{\bar{\sigma}} \xi^{\bar{\delta}} \rangle = \Delta_0 (\mathcal{G}\mathcal{G} - \mathcal{L}\mathcal{L})^{(\bar{\alpha}\bar{\sigma}\bar{\beta}\bar{\delta} - \bar{\alpha}\bar{\delta}\bar{\beta}\bar{\sigma})}, \quad (104)$$

$$\langle \xi^{\bar{\alpha}} \partial_1 \xi^{\bar{\beta}} \partial_0 \xi^{\bar{\sigma}} \xi^{\bar{\delta}} \rangle = -\Delta_0 (\mathcal{G}\mathcal{L} + \mathcal{L}\mathcal{G})^{(\leftrightarrow)} - \theta \mathcal{L}\mathcal{L}^{(\leftrightarrow)}, \quad (105)$$

$$\langle \xi^{\bar{\alpha}} \partial_0 \xi^{\bar{\beta}} \partial_0 \xi^{\bar{\sigma}} \xi^{\bar{\delta}} \rangle = -\Delta_0 (\mathcal{G}\mathcal{G} + 3\mathcal{L}\mathcal{L})^{(\leftrightarrow)} - \theta (\mathcal{L}\mathcal{G} + \mathcal{G}\mathcal{L})^{(\leftrightarrow)}, \quad (106)$$

where $\mathcal{G}\mathcal{G}^{(\bar{\alpha}\bar{\sigma}\bar{\beta}\bar{\delta} - \bar{\alpha}\bar{\delta}\bar{\beta}\bar{\sigma})}$ represents $\mathcal{G}^{\bar{\alpha}\bar{\sigma}}\mathcal{G}^{\bar{\beta}\bar{\delta}} - \mathcal{G}^{\bar{\alpha}\bar{\delta}}\mathcal{G}^{\bar{\beta}\bar{\sigma}}$ and (\leftrightarrow) understood in the same way.

¹²We will simplify notation by using $\langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = i \int d^2 \sigma' \langle \xi(\sigma)^{\bar{A}} \partial_1 \xi(\sigma)^{\bar{B}} \xi(\sigma')^{\bar{C}} \partial_0 \xi(\sigma')^{\bar{D}} \rangle$ it will always appear in the expansion of the exponential of the effective action in this form.

References

- [1] C. M. Hull, “Global aspects of T-duality, gauged sigma models and T-folds,” arXiv:hep-th/0604178.
A. Dabholkar and C. Hull, “Generalised T-duality and non-geometric backgrounds,” JHEP **0605** (2006) 009 [arXiv:hep-th/0512005].
J. Shelton, W. Taylor and B. Wecht, “Nongeometric flux compactifications,” JHEP **0510** (2005) 085 [arXiv:hep-th/0508133].
J. Shelton, W. Taylor and B. Wecht, “Generalized flux vacua,” arXiv:hep-th/0607015.
C. M. Hull and R. A. Reid-Edwards, “Flux compactifications of string theory on twisted tori,” arXiv:hep-th/0503114.
K. Becker, M. Becker, C. Vafa and J. Walcher, “Moduli stabilization in non-geometric backgrounds,” arXiv:hep-th/0611001.
- [2] M. J. Duff, “Duality Rotations In String Theory,” Nucl. Phys. B **335** (1990) 610. J. H. Schwarz and A. Sen, “Duality symmetric actions,” Nucl. Phys. B **411** (1994) 35 [arXiv:hep-th/9304154].
- [3] C. M. Hull, “A geometry for non-geometric string backgrounds,” JHEP **0510** (2005) 065 [arXiv:hep-th/0406102].
- [4] C. M. Hull, “Doubled geometry and T-folds,” arXiv:hep-th/0605149.
- [5] A. A. Tseytlin, “Duality Symmetric Formulation Of String World Sheet Dynamics,” Phys. Lett. B **242** (1990) 163.
- [6] A. A. Tseytlin, “Duality Symmetric Closed String Theory And Interacting Chiral Scalars,” Nucl. Phys. B **350** (1991) 395.
- [7] D. S. Berman and N. B. Copland, “The string partition function in Hull’s doubled formalism,” Phys. Lett. B **649** (2007) 325 [arXiv:hep-th/0701080].

- [8] E. Hackett-Jones and G. Moutsopoulos, “Quantum mechanics of the doubled torus,” JHEP **0610** (2006) 062 [arXiv:hep-th/0605114].
- [9] S. P. Chowdhury, “Superstring partition functions in the doubled formalism,” arXiv:0707.3549 [hep-th].
- [10] R. Gregory, J. A. Harvey and G. W. Moore, “Unwinding strings and T-duality of Kaluza-Klein and H-monopoles,” Adv. Theor. Math. Phys. **1** (1997) 283 [arXiv:hep-th/9708086].
J. A. Harvey and S. Jensen, “Worldsheet instanton corrections to the Kaluza-Klein monopole,” JHEP **0510** (2005) 028 [arXiv:hep-th/0507204].
- [11] D. Tong, “NS5-branes, T-duality and worldsheet instantons,” JHEP **0207** (2002) 013 [arXiv:hep-th/0204186].
- [12] P. Pasti, D. P. Sorokin and M. Tonin, “On Lorentz invariant actions for chiral p-forms,” Phys. Rev. D **55** (1997) 6292 [arXiv:hep-th/9611100].
- [13] R. Floreanini and R. Jackiw, “Selfdual Fields As Charge Density Solitons,” Phys. Rev. Lett. **59** (1987) 1873.
- [14] J. Honerkamp, “Chiral multiloops,” Nucl. Phys. B **36** (1972) 130.
L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, “The Background Field Method And The Ultraviolet Structure Of The Supersymmetric Nonlinear Sigma Model,” Annals Phys. **134** (1981) 85.
- [15] E. Braaten, T. L. Curtright and C. K. Zachos, “Torsion And Geometrostasis In Nonlinear Sigma Models,” Nucl. Phys. B **260** (1985) 630.
- [16] S. Mukhi, “The Geometric Background Field Method, Renormalization And The Wess-Zumino Term In Nonlinear Sigma Models,” Nucl. Phys. B **264** (1986) 640.
- [17] C. G. Callan, E. J. Martinec, M. J. Perry and D. Friedan, “Strings In Background Fields,” Nucl. Phys. B **262**, 593 (1985).

- [18] A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory,”
Phys. Rept. **244** (1994) 77 [arXiv:hep-th/9401139].